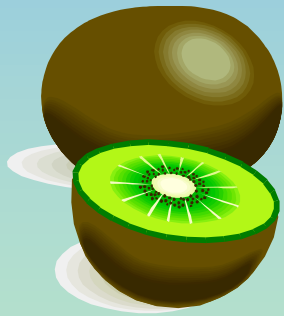# Fractions

$^{55}/_{60}$

$^1/_8$

$^{11}/_{12}$

$1\ ^2/_{10}$

$1\ ½$

$^1/_{12}$

What is a fraction?

Loosely speaking, a fraction is a quantity that cannot be represented by a whole number.

Why do we need fractions?

Consider the following scenario.

Can you finish the whole cake?

If not, how many cakes did you eat?
1 is not the answer, neither is 0.

This suggest that we need a new kind of number.

Definition:

A fraction is an ordered pair of whole numbers, the 1st one is usually written on top of the other, such as ½ or ¾ .

$$\frac{a}{b}$$

⟵ numerator

⟵ denominator

The **denominator** tells us how many congruent pieces the whole is divided into, thus this number cannot be 0.

The **numerator** tells us how many such pieces are being considered.

Examples:

How much of a pizza do we have below?

- we first need to know the size of the original pizza.

The blue circle is our <u>whole</u>.
- if we divide the whole into 8 congruent pieces,
- the denominator would be **8**.

We can see that we have 7 of these pieces.
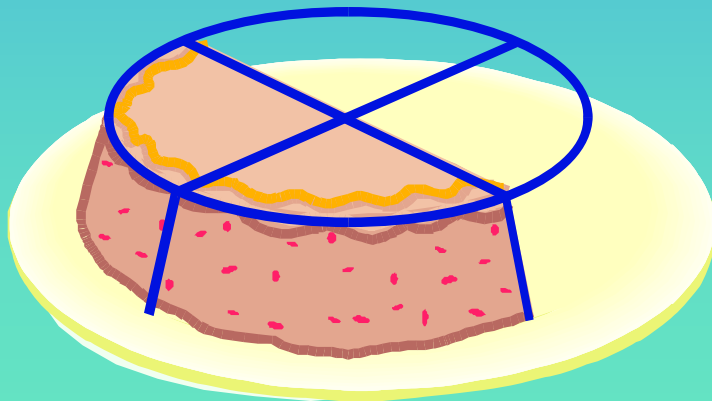Therefore the numerator is **7**, and we have

$$\frac{7}{8}$$ of a pizza.

Equivalent fractions

a fraction can have many different appearances, these are called <u>equivalent</u> fractions

In the following picture we have **½** of a cake because the whole cake is divided into two congruent parts and we have only one of those parts.

But if we cut the cake into smaller congruent pieces, we can see that

$$\frac{1}{2} = \frac{2}{4}$$

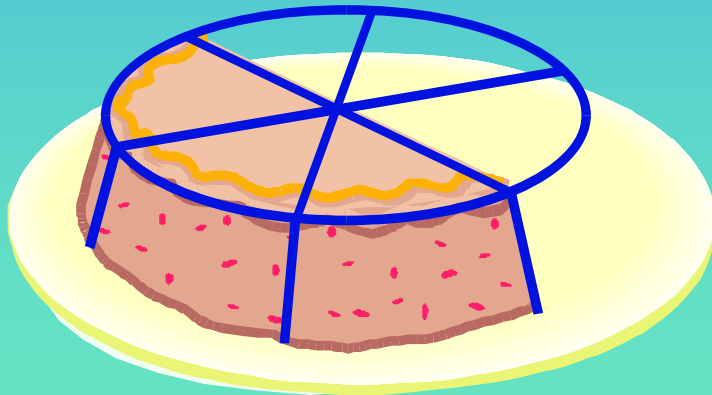Or we can cut the original cake into 6 congruent pieces,

Equivalent fractions

a fraction can have many different appearances, these are called <u>equivalent</u> fractions

Now we have 3 pieces out of 6 equal pieces, but the total amount we have is still the same.

Therefore,

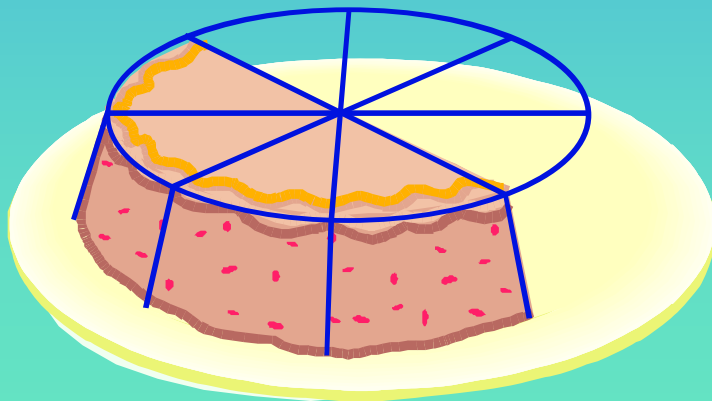$$\frac{1}{2} \; = \; \frac{2}{4} \; = \; \frac{3}{6}$$

If you don't like this, we can cut the original cake into 8 congruent pieces,

Equivalent fractions

a fraction can have many different appearances, they are called <u>equivalent</u> fractions

then we have 4 pieces out of 8 equal pieces, but the total amount we have is still the same.

Therefore,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$$

We can generalize this to

$$\frac{1}{2} = \frac{1 \times n}{2 \times n}$$ whenever n is not 0

How do we know that two fractions are the same?

we cannot tell whether two fractions are the same until we reduce them to their lowest terms.

A fraction is in its <u>lowest terms</u> (or is <u>reduced</u>) if we cannot find a whole number (other than 1) that can divide into both its numerator and denominator.

Examples:

$\dfrac{6}{10}$ is not reduced because 2 can divide into both 6 and 10.

$\dfrac{35}{40}$ is not reduced because 5 divides into both 35 and 40.

How do we know that two fractions are the same?

More examples:

$\dfrac{110}{260}$ is not reduced because 10 can divide into both 110 and 260.

$\dfrac{8}{15}$ is reduced.

$\dfrac{11}{23}$ is reduced

To find out whether two fraction are equal, we need to reduce them to their lowest terms.

How do we know that two fractions are the same?

Examples:

Are $\dfrac{14}{21}$ and $\dfrac{30}{45}$ equal?

$\dfrac{14}{21} \xrightarrow{\textit{reduce}} \dfrac{14 \div 7}{21 \div 7} = \dfrac{2}{3}$

$\dfrac{30}{45} \xrightarrow{\textit{reduce}} \dfrac{30 \div 5}{45 \div 5} = \dfrac{6}{9} \xrightarrow{\textit{reduce}} \dfrac{6 \div 3}{9 \div 3} = \dfrac{2}{3}$

Now we know that these two fractions are actually the same!

How do we know that two fractions are the same?

Another example:

Are $\dfrac{24}{40}$ and $\dfrac{30}{42}$ equal?

$$\dfrac{24}{40} \xrightarrow{\;reduce\;} \dfrac{24 \div 2}{40 \div 2} = \dfrac{12}{20} \xrightarrow{\;reduce\;} \dfrac{12 \div 4}{20 \div 4} = \dfrac{3}{5}$$

$$\dfrac{30}{42} \xrightarrow{\;reduce\;} \dfrac{30 \div 6}{42 \div 6} = \dfrac{5}{7}$$

This shows that these two fractions are not the same!

Improper Fractions and Mixed Numbers

An *improper fraction* is a fraction with the *numerator* larger than or equal to the *denominator*.

$$\frac{5}{3}$$

Any whole number can be transformed into an improper fraction.

$$4 = \frac{4}{1}, \quad 1 = \frac{7}{7}$$

A *mixed number* is a whole number and a fraction together

$$2\frac{3}{7}$$

An improper fraction can be converted to a mixed number and vice versa.

Improper Fractions and Mixed Numbers

Converting improper fractions into mixed numbers:
- *divide the numerator by the denominator*
- *the quotient is the leading number,*
- *the remainder as the new numerator.*

$$\frac{5}{3} = 1\frac{2}{3}$$

More examples:

$$\frac{7}{4} = 1\frac{3}{4}, \quad \frac{11}{5} = 2\frac{1}{5}$$

Converting mixed numbers into improper fractions.

$$2\frac{3}{7} = \frac{2 \times 7 + 3}{7} = \frac{17}{7}$$

How does the denominator control a fraction?

If you share a pizza evenly among two people, you will get $\dfrac{1}{2}$
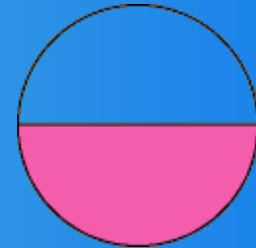
If you share a pizza evenly among three people, you will get $\dfrac{1}{3}$
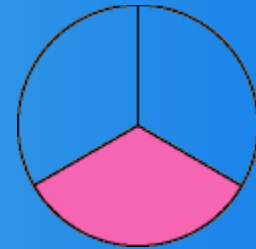
If you share a pizza evenly among four people, you will get $\dfrac{1}{4}$

How does the denominator control a fraction?

If you share a pizza evenly among eight people, you will get only $\dfrac{1}{8}$
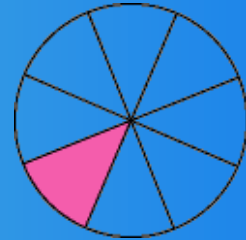
It's not hard to see that the slice you get becomes smaller and smaller.

Conclusion:

The larger the denominator the smaller the pieces, and if the numerator is kept fixed, the larger the denominator the <u>smaller</u> the fraction,

$$\text{i.e.} \quad \frac{a}{b} < \frac{a}{c} \quad \text{whenever} \quad b > c.$$

Examples:

Which one is larger, $\dfrac{2}{7}$ or $\dfrac{2}{5}$?　　　　Ans: $\dfrac{2}{5}$

Which one is larger, $\dfrac{8}{23}$ or $\dfrac{8}{25}$?　　　　Ans: $\dfrac{8}{23}$

Which one is larger, $\dfrac{41}{135}$ or $\dfrac{41}{267}$?　　　　Ans: $\dfrac{41}{135}$

How does the numerator affect a fraction?

Here is $^1/_{16}$,
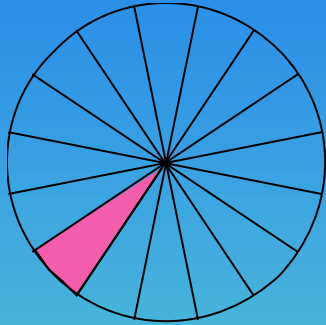
here is $^3/_{16}$,
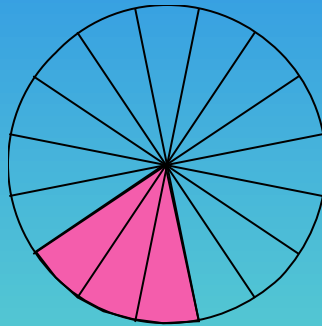
here is $^5/_{16}$,
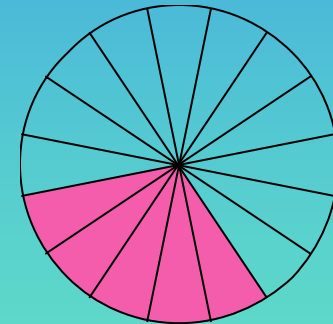
Do you see a trend?
Yes, when the numerator gets larger
we have more pieces.
And if the denominator is kept fixed,
the <u>larger</u> numerator makes a <u>bigger</u>
fraction.

Examples:

Which one is larger, $\dfrac{7}{12}$ or $\dfrac{5}{12}$? Ans : $\dfrac{7}{12}$

Which one is larger, $\dfrac{8}{20}$ or $\dfrac{13}{20}$? Ans : $\dfrac{13}{20}$

Which one is larger, $\dfrac{45}{100}$ or $\dfrac{63}{100}$? Ans : $\dfrac{63}{100}$

Comparing fractions with different numerators and different denominators.

In this case, it would be pretty difficult to tell just from the numbers which fraction is bigger, for example

$$\frac{3}{8}$$

$$\frac{5}{12}$$
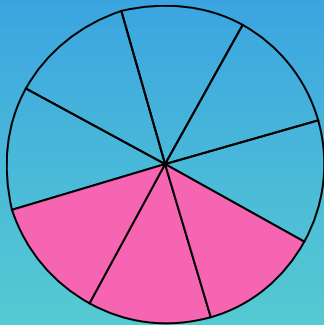
This one has less pieces but each piece is larger than those on the right.
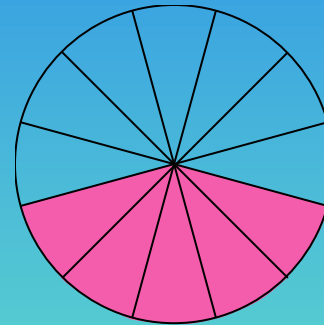
This one has more pieces but each piece is smaller than those on the left.
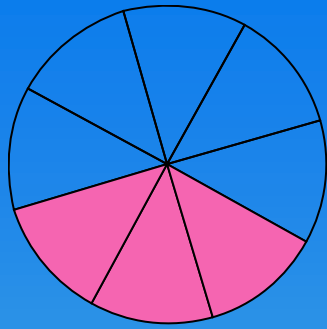
$$\frac{3}{8}$$
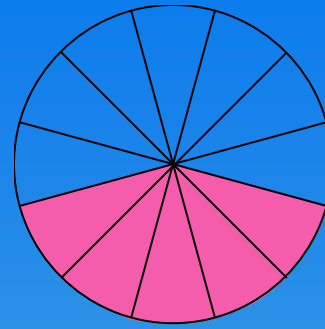
$$\frac{5}{12}$$

One way to answer this question is to change the appearance of the fractions so that the denominators are the same.

In that case, the pieces are all of the same size, hence the larger numerator makes a bigger fraction.

The straight forward way to find a <u>common denominator</u> is to multiply the two denominators together:

$$\frac{3}{8} = \frac{3 \times 12}{8 \times 12} = \frac{36}{96} \qquad \text{and} \qquad \frac{5}{12} = \frac{5 \times 8}{12 \times 8} = \frac{40}{96}$$

Now it is easy to tell that 5/12 is actually a bit bigger than 3/8.

A more efficient way to compare fractions

Which one is larger,

$$\frac{7}{11} \text{ or } \frac{5}{8} \text{ ?}$$

From the previous example, we see that we don't really have to know what the common denominator turns out to be, all we care are the *numerators*. Therefore we shall only change the numerators by cross multiplying.

$$\frac{7}{11} \times \frac{5}{8}$$

$$7 \times 8 = 56 \qquad\qquad 11 \times 5 = 55$$

Since $56 > 55$, we see that $\quad \dfrac{7}{11} > \dfrac{5}{8}$

This method is called cross-multiplication, and make sure that you remember to make the arrows go upward.

Addition of Fractions

- addition means combining objects in two or more sets
- the objects must be of the same type, i.e. we combine bundles with bundles and sticks with sticks.
- in fractions, we can only combine pieces of the same size. In other words, the denominators must be the same.

Addition of Fractions with equal denominators

Example: $\dfrac{1}{8} + \dfrac{3}{8}$

$+$ $= \ ?$
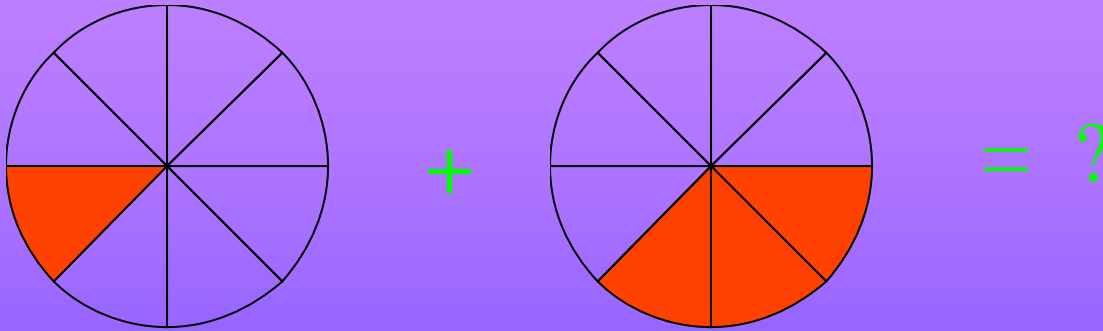
Click to see animation

Addition of Fractions with equal denominators
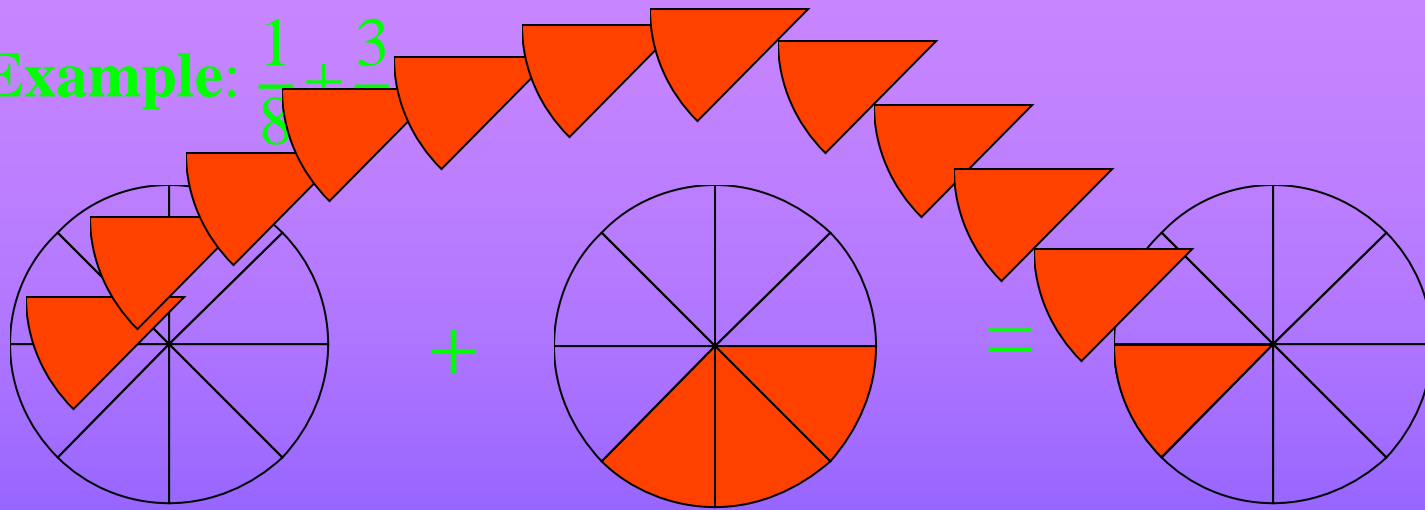
Example: $\dfrac{1}{8} + \dfrac{3}{8}$

Addition of Fractions with equal denominators

Example: $\dfrac{1}{8} + \dfrac{3}{8}$
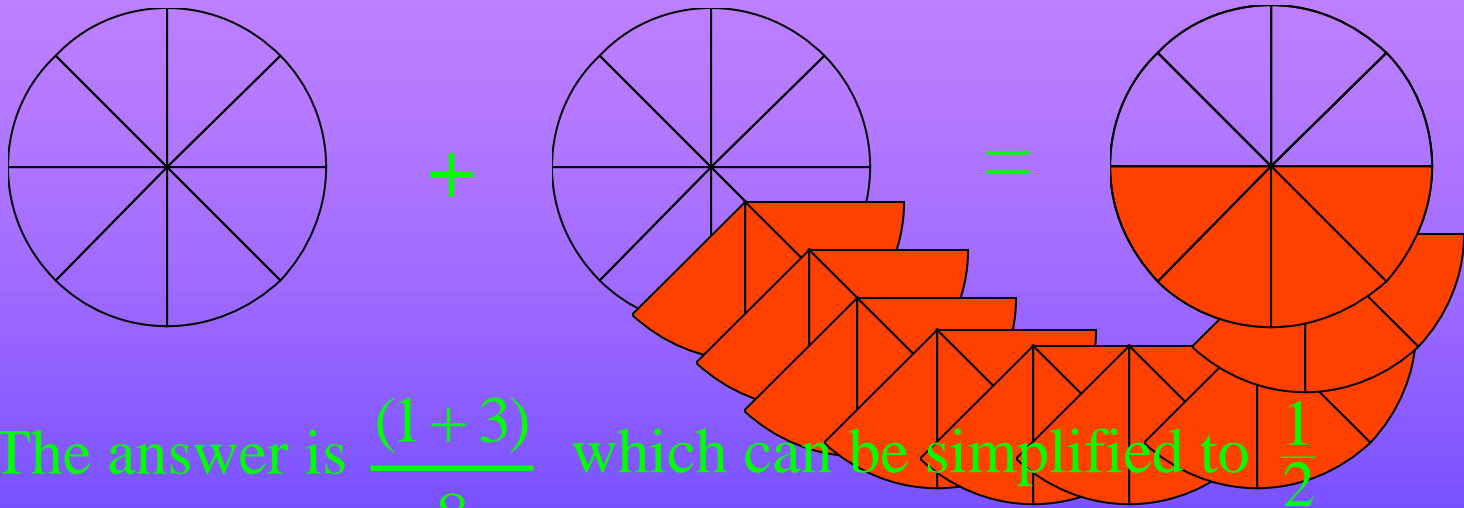
The answer is $\dfrac{(1+3)}{8}$ which can be simplified to $\dfrac{1}{2}$

$\dfrac{(1+3)}{(8+8)}$ is NOT the right answer because the denominator tells us how many pieces the whole is divided into, and in this addition problem, we have not changed the number of pieces in the whole. Therefore the denominator should still be **8**.

Addition of Fractions with equal denominators

More examples

$$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

$$\frac{6}{10} + \frac{7}{10} = \frac{13}{10} = 1\frac{3}{10}$$

$$\frac{6}{15} + \frac{8}{15} = \frac{14}{15}$$

Addition of Fractions with different denominators

In this case, we need to first convert them into equivalent fraction with the same denominator.

Example:
$$\frac{1}{3} + \frac{2}{5}$$

An easy choice for a common denominator is $3 \times 5 = 15$

$$\frac{1}{3} = \frac{1 \times 5}{3 \times 5} = \frac{5}{15} \qquad \frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$$

Therefore,
$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$$

Addition of Fractions with
different denominators

Remark: When the denominators are bigger, we need to find the least common denominator by factoring.

If you do not know prime factorization yet, you have to multiply the two denominators together.

More Exercises:

$$\frac{3}{4} + \frac{1}{8} = \frac{3 \times 2}{4 \times 2} + \frac{1}{8} = \frac{6}{8} + \frac{1}{8} = \frac{6+1}{8} = \frac{7}{8}$$

$$\frac{3}{5} + \frac{2}{7} = \frac{3 \times 7}{5 \times 7} + \frac{2 \times 5}{7 \times 5} = \frac{21}{35} + \frac{10}{35} = \frac{21+10}{35} = \frac{31}{35}$$

$$\frac{5}{6} + \frac{4}{9} = \frac{5 \times 9}{6 \times 9} + \frac{4 \times 6}{9 \times 6} = \frac{45}{54} + \frac{24}{54}$$

$$= \frac{45+24}{54} = \frac{69}{54} = 1\frac{15}{54}$$

Adding Mixed Numbers

Example:

$$3\frac{1}{5} + 2\frac{3}{5} = 3 + \frac{1}{5} + 2 + \frac{3}{5}$$

$$= 3 + 2 + \frac{1}{5} + \frac{3}{5}$$

$$= 5 + \frac{1+3}{5}$$

$$= 5 + \frac{4}{5}$$

$$= 5\frac{4}{5}$$

Adding Mixed Numbers

Another Example:

$$2\frac{4}{7}+1\frac{3}{8} = 2+\frac{4}{7}+1+\frac{3}{8}$$

$$= 2+1+\frac{4}{7}+\frac{3}{8}$$

$$= 3+\frac{4\times8+3\times7}{56}$$

$$= 3+\frac{53}{56} = 3\frac{53}{56}$$

Subtraction of Fractions

- subtraction means taking objects away.
- the objects must be of the same type, i.e. we can only take away apples from a group of apples.
- in fractions, we can only take away pieces of the same size. In other words, the denominators must be the same.

Subtraction of Fractions with _equal_ denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$
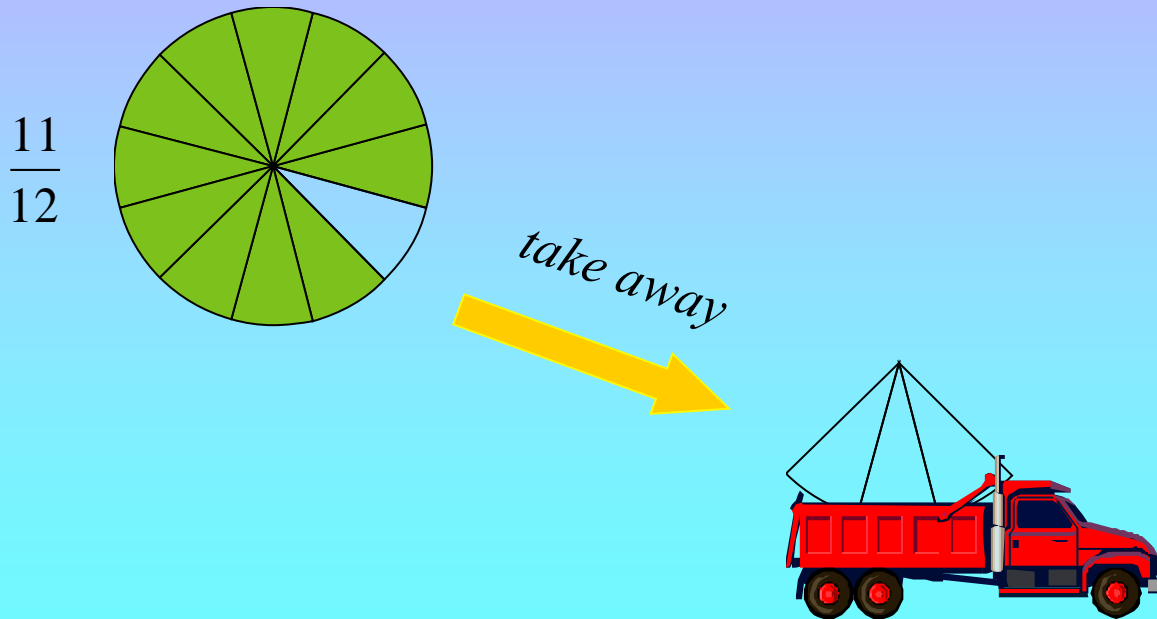
$\dfrac{11}{12}$

take away

(Click to see animation)

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

$\dfrac{11}{12}$

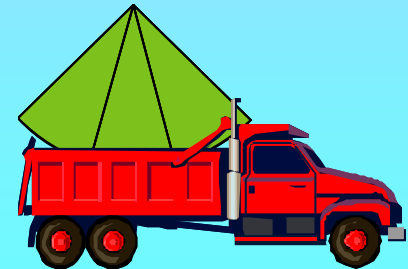# Subtraction of Fractions with equal denominators

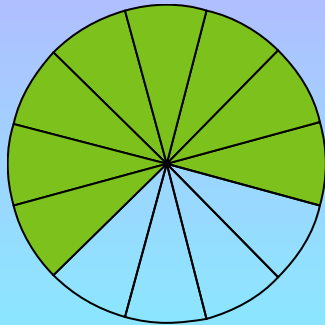Example: $\dfrac{11}{12} - \dfrac{3}{12}$

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Subtraction of Fractions with <u>equal</u> denominators

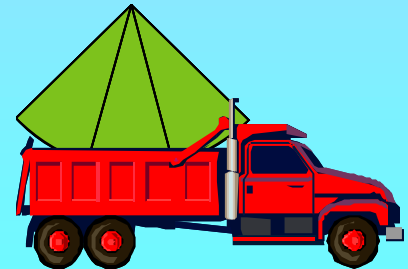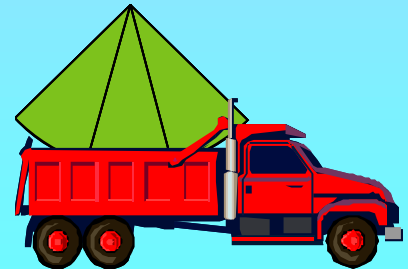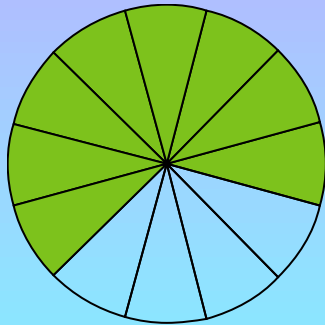Example: $\dfrac{11}{12} - \dfrac{3}{12}$

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Subtraction of Fractions with _equal_ denominators

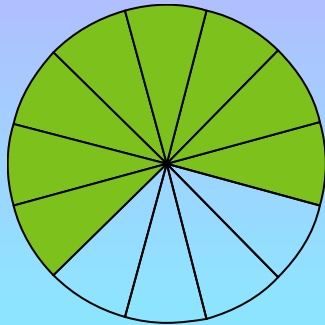Example: $\dfrac{11}{12} - \dfrac{3}{12}$

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Subtraction of Fractions with equal denominators
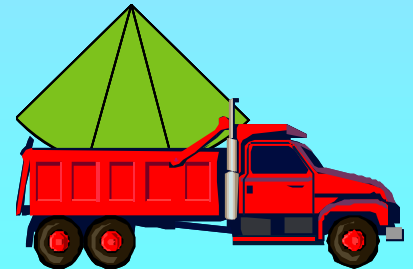
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators
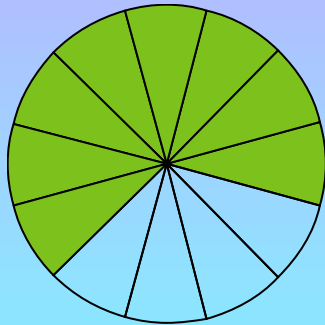
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators
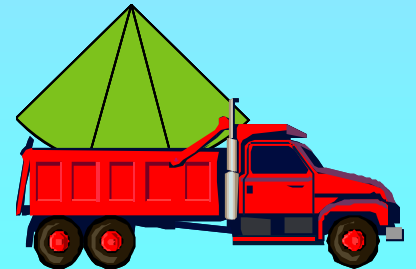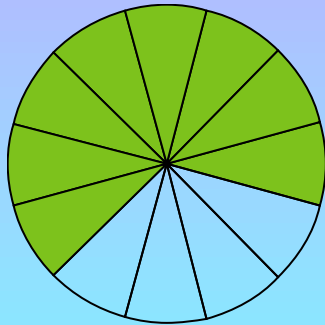
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

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Subtraction of Fractions with equal denominators
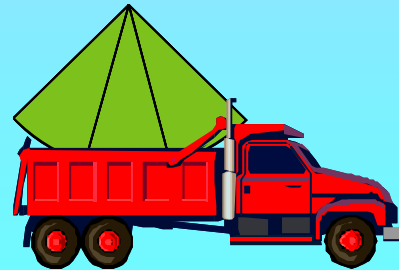
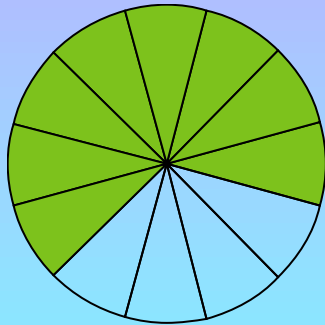Example: $\dfrac{11}{12} - \dfrac{3}{12}$

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Subtraction of Fractions with <u>equal</u> denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with <u>equal</u> denominators
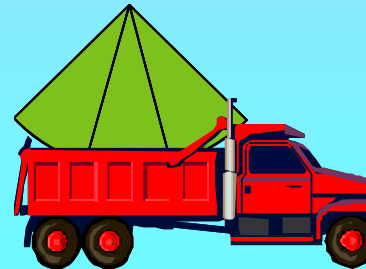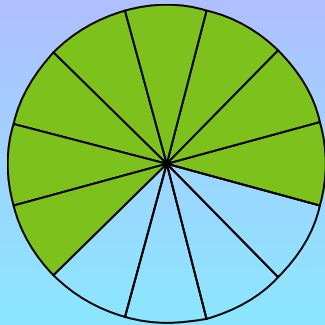
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators
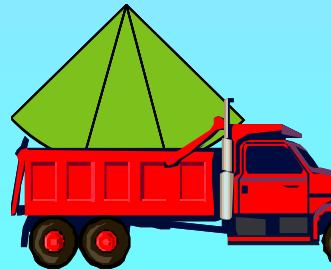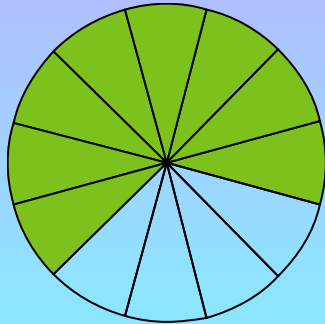
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with **equal** denominators

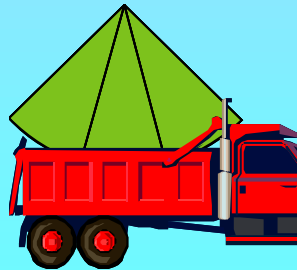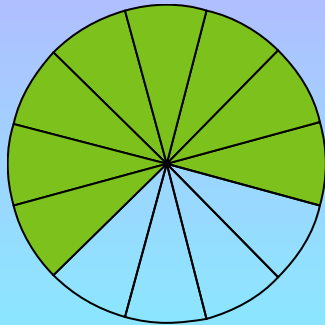Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

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Subtraction of Fractions with equal denominators
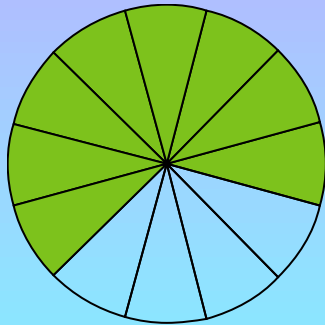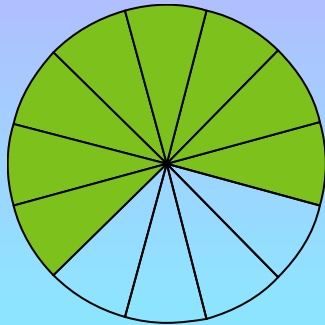
Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

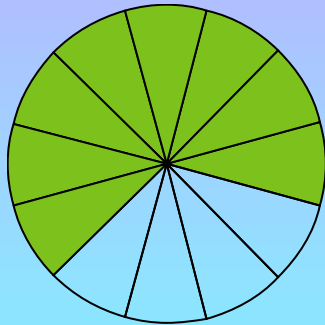Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$

This means to take away $\dfrac{3}{12}$ from $\dfrac{11}{12}$

Subtraction of Fractions with equal denominators

Example: $\dfrac{11}{12} - \dfrac{3}{12}$
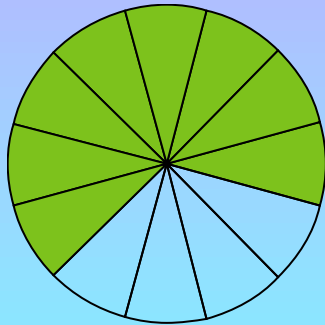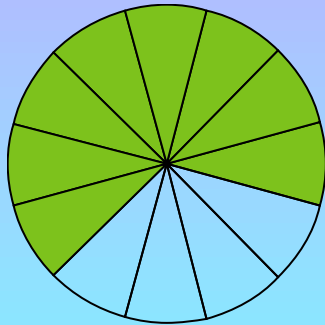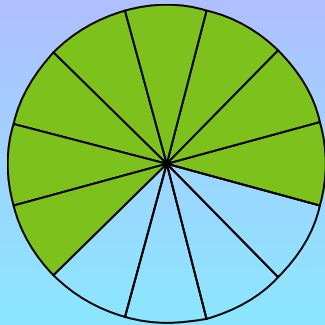
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Now you can see that there are only 8 pieces left, therefore

$$\dfrac{11}{12} - \dfrac{3}{12} = \dfrac{11-3}{12} = \dfrac{8}{12} = \dfrac{2}{3}$$

Subtraction of Fractions

More examples:

$$\frac{15}{16} - \frac{7}{16} = \frac{15-7}{16} = \frac{8}{16} = \frac{1}{2}$$

$$\frac{6}{7} - \frac{4}{9} = \frac{6 \times 9}{7 \times 9} - \frac{4 \times 7}{9 \times 7} = \frac{54}{63} - \frac{28}{63} = \frac{54-28}{63} = \frac{26}{63}$$

$$\frac{7}{10} - \frac{11}{23} = \frac{7 \times 23}{10 \times 23} - \frac{11 \times 10}{23 \times 10} = \frac{7 \times 23 - 11 \times 10}{10 \times 23} = \frac{161-110}{230}$$

$$= \frac{51}{230}$$

Did you get all the answers right?

Subtraction of mixed numbers

This is more difficult than before, so please take notes.

Example: $3\dfrac{1}{4} - 1\dfrac{1}{2}$

Since 1/4 is not enough to be subtracted by 1/2, we better convert all mixed numbers into improper fractions first.

$$3\dfrac{1}{4} - 1\dfrac{1}{2} = \dfrac{3\times 4 + 1}{4} - \dfrac{1\times 2 + 1}{2}$$

$$= \dfrac{13}{4} - \dfrac{3}{2}$$

$$= \dfrac{13}{4} - \dfrac{6}{4}$$

$$= \dfrac{7}{4} = 1\dfrac{3}{4}$$